Journal of Geophysics and Engineering

PAPER

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IOP Publishing Journal of Geophysics and Engineering
J. Geophys. Eng. 15 (2018) 2686–2694 (9pp) https://doi.org/10.1088/1742-2140/aae3be

Application of the power mean to modeling the elastic properties of reservoir rocks

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Received 1 June 2018, revised 7 September 2018
Accepted for publication 24 September 2018
Published 25 October 2018

Abstract

The relationship between seismic velocity and porosity is one of the most important research topics in exploration geophysics. A strong correlation, if it exists, can be very useful for quantitative seismic interpretation of direct hydrocarbon indicators. The Voigt and Reuss bounds are often plotted alongside the data for quality control and interpretation. The relative position of the data within the bounds may supply important information regarding the pore shape, stress condition, lithology, and diagenesis of porous rocks. It is found that the Voigt bound and Reuss bound are actually special cases of the weighted power mean, with the power parameters being 1 and −1, respectively. The geometric mean is a special case of the power mean with the power parameter being zero. The power mean is a monotonic function of the power parameter. For the construction of a rock physics template, the region between the Voigt and Reuss bounds can be evenly divided by varying the power parameter between −1 and 1. The power mean can be a very useful tool for modeling the elastic properties of rocks. We have demonstrated the potential applications of the power mean in the studies of velocity-porosity relation, effective media, and pore fluids effect on seismic velocities.

Keywords: power mean, effective medium, geometric mean, Voigt bound, Reuss bound

Introduction

Discovering the relationship between the seismic velocities and porosity is one of the most important topics in rock physics studies (Nur *et al* 1995). The seismic velocities, which are functions of the elastic properties and bulk density, are affected by other properties of porous rock, such as the pore geometry, texture, and heterogeneity of the rocks. The pore geometry refers to the distribution of pores of various shapes and dimensions, connectivity, and surface-to-volume ratio. The pore geometry can have a significant effect on the seismic velocities of porous rocks (Wang *et al* 2015). The seismic velocities are also stress-dependent. Therefore, it is very difficult to build a theoretical model to describe the relationship between the seismic velocities and porosity.

Most of the previous works on the velocity-porosity relationship are based on empirical relations (Wyllie *et al* 1956, Raymer *et al* 1980, Tosaya and Nur 1982, Han 1986, Xu and White 1995), and other parameters, such as the mineral content and stress, may be included in the empirical relation. For a rock with a certain porosity, the elastic bounds give the possible range of the elastic properties. It is common practice to plot the bounds with the measurement data to observe the position and trends of the data relative to the bounds. The relative position may supply important information regarding the pore shape, diagenesis, pore fluid type and pressure conditions (Marion and Nur 1991, Mavko and Mukerji 1995, Fabricius *et al* 2008, Yan and Han 2011, Zhao *et al* 2013). The best-known elastic bounds are the Voigt (1928) upper bound and the Reuss (1929) lower bound due to their simplicity in formulation and theoretical validity. The separation of Hashin–Shtrikman (1963) bounds is usually much narrower but they take more complicated forms than the Voigt–Reuss bounds. Nur *et al* (1995) brought up the concept of 'critical porosity' to modify the Voigt–Reuss bounds to better confine the data for natural porous rocks. The Hashin–Shtrikman bounds can also be modified using 'critical porosity' (Fabricius 2003), but some valid data might fall out of the modified bounds because the original bounds are already narrow at the low porosity range.

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The region between the bounds is often equally divided in the *y*-axis direction to mark the relative position; the divisional curves are called 'iso-frame' curves by Fabricius (2003) because the rocks located on the iso-frame curve are believed to have the same induration or hardness.

Nur *et al* (1995) noticed that there is a mathematical similarity between the Voigt bound and Reuss bound. In this study, we further explored the mathematical similarity and found that the Voigt bound and Reuss bounds are special forms of the weighted power mean in the terminology of statistics. They can be represented by a common and simple formula with a power parameter denoting the relative hardness of a rock. The power mean is the Voigt bound when the power parameter is 1 and it is the Reuss bound when the power parameter is −1. The power mean is a monotonic function of the power parameter. We will demonstrate that the power mean, with the power parameter varying in the range from −1 to 1, can be a powerful tool in rock physics modeling.

The power mean and the bounds

Given a set of values (M_1, M_2, …, M_N), and the corresponding weights (f_1, f_2, …, f_N, and with $\sum_{n=1}^{N} f_n = 1$), the weighted power mean (Bullen 2003) can be written as

$$M = (f_1 M_1{}^a + f_1 M_1{}^a + \ldots + f_N M_N{}^a)^{1/a}. \quad (1)$$

Here a is the power parameter. In the study of the effective properties of a rock, the power mean should always be weighted by the proportions of finite phases or compositions of the rock. For convenience of reference, we call the weighted power mean the power mean in this study. A power mean is also known as a general mean, Hölder mean, or mean of degree. Some well-known means are special cases of the general mean when a takes special values. When a takes values of −1, 0, 1, and 2, the general mean becomes the harmonic mean, geometric mean, arithmetic mean, and root-mean-square, respectively. The geometric mean takes a different form from the general mean,

$$M = M_1{}^{f_1} M_2{}^{f_2} \ldots M_N{}^{f_N} = \prod_{n=1}^{N} M_n{}^{f_n}. \quad (2)$$

The specialization of the geometric mean from the general mean is not straightforward. It can also be numerically tested by parameterizing equation (1) with an infinitesimally small value of a.

The propagation speeds of acoustic waves are determined by the effective elastic properties and density of the composite rock, which may consist of different minerals and pore fluids. In studying the effective elastic properties of the rocks, the M variable in equations (1) and (2) can be bulk modulus (K), shear modulus (G), P-wave modulus (M), etc. The weights f are the volume fraction of the constituent phases of a rock. In such cases, the weighted harmonic mean is actually the well-known Reuss bound, and the weighted arithmetic mean is the Voigt bound. This point is self-evident when only two phases, the solid frame and pore fluid, are considered:

$$K_{wet} = ((1 - \phi)K_m{}^a + \phi K_f{}^a)^{1/a}, \quad (3)$$

where K_m and K_f are the bulk modulus of the solid frame and the pore fluid, respectively, ϕ is the porosity and K_{wet} is the effective bulk modulus of the saturated rock. When a is set equal to −1 or 1, the aforementioned expression becomes the Reuss bound and the Voigt bound, respectively,

$$K_{wet}{}^{-1} = (1 - \phi)K_m{}^{-1} + \phi K_f{}^{-1}, \quad (4)$$

$$K_{wet} = (1 - \phi)K_m + \phi K_f\ . \quad (5)$$

The mathematical similarity between the Voigt bound and Reuss bound is obvious when the bulk modulus terms are written in exponential forms as shown in equation (4). When a goes to 0, equation (3) becomes the geometric mean

$$K_{wet} = K_m{}^{1-\phi} K_f{}^{\phi}. \quad (6)$$

One important feature of the power mean is that it is a monotonic function of the power parameter a for a given set of unequal real numbers, M_1, M_2, …, M_N (Qi *et al* 2000, Bullen 2003, Witkowski 2004a, 2004b). In rock physics, $a = -1$ and $a = 1$ correspond to the lower Reuss bound and the upper Voigt bound of the effective elastic properties of a rock. For real rocks the power parameter a should be dependent on rock texture, diagenesis and stress history, and vary within the range of [−1, 1]. It should be an important parameter to characterize the elastic properties of the rocks. Rock physics data are often plotted with the Voigt bound and Reuss bound, or the Hashin–Shtrikman bounds (Hashin and Shtrikman 1963). Inside the bounds, curves with equal vertical divisions between the bounds are plotted to mark the relative positions of the data. These curves are called iso-frame curves by Fabricius (2003); they are a measure of induration (or hardness) for rocks with a given porosity. We call these curves iso-fraction curves because they are simply equal fractional divisions between the bounds. Similar curves can be built using equation (1) by letting a vary from −1 to 1. These curves are called iso-power curves.

Figure 1 illustrates the relationship between the rock properties, the iso-fraction curves, and the iso-power curves. The rock physics data are from laboratory measurements by Han (1986), and the color bar denotes the clay content of the sandstone samples. We decide to use the P-wave modulus since P-wave velocity is predominately used in seismic data processing and interpretation, and the two-term power mean can be written in the form

$$M_{wet} = ((1 - \phi)M_m{}^{a_{wet}} + \phi M_f{}^{a_{wet}})^{1/a_{wet}}, \quad (7)$$

$$M_{wet} = M_m{}^{1-\phi} M_f{}^{\phi},\ (a_{wet} = 0). \quad (8)$$

Here M refers to the P-wave bulk modulus and a_{wet} denotes the power parameter for fully brine-saturated rock. The iso-frame curves in figure 1 are based on the elastic moduli of quartz and water given in table 1. For the region enclosed between the Voigt bound and Reuss bound, the iso-power curves are more evenly distributed than the iso-fraction curves in the direction perpendicular to the long axis of the bounded area. Some of the iso-fraction curves are packed together in

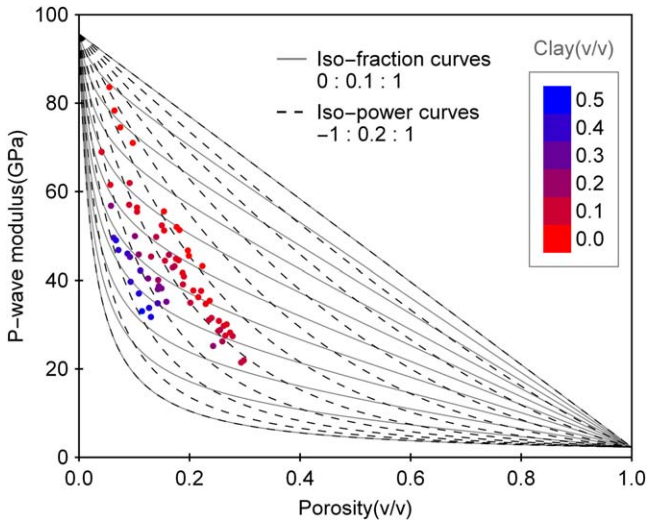

Figure 1. Comparison of the iso-fraction curves and iso-power curves with Han's data (Han 1986). −1:0.2:1 denotes that the value increases from −1 to 1 by steps of 0.2.

Table 1. Mineral elastic moduli and density used in this study (Referred to Mavko *et al* 1998).

	Quartz	Calcite	Clay	Dolomite	Water
K (GPa)	37	75	25	78	2.3
G (GPa)	44	30	9	48	0
ρ_m (g cc^{-1})	2.65	2.71	2.65	2.87	1.02

the lower porosity part. This feature is also obvious when the Hashin–Shtrikman bounds and critical porosity are used to construct the iso-fraction curves, as can be seen from figure 2 by Fabricius (2003). The trending of the iso-power curves is generally more congruous with the data trend, especially when rocks with different lithology are considered separately. Apart from the simplicity of their mathematical form, the power means with $a = -1$, 0, 1 also have distinct physical meaning relating to the strain-stress distribution of the composite material (Mainprice and Humbert 1994). Therefore, the power mean with power parameters in the range $[-1, 1]$ may be more suitable for building rock physics templates and for modeling the elastic properties of porous rocks, and the power parameter or iso-power curves are better indications of the rock hardness than the iso-fraction curves.

Modeling the velocity-porosity relationship

Finding the relationship between porosity and seismic wave velocities in porous rocks has been an important area of research for decades. Wyllie *et al* (1956, 1958) revealed that a simple relation exists between the velocity and porosity in sedimentary rocks when the fluid-saturated rocks have uniform mineralogy and are at high effective pressure. This

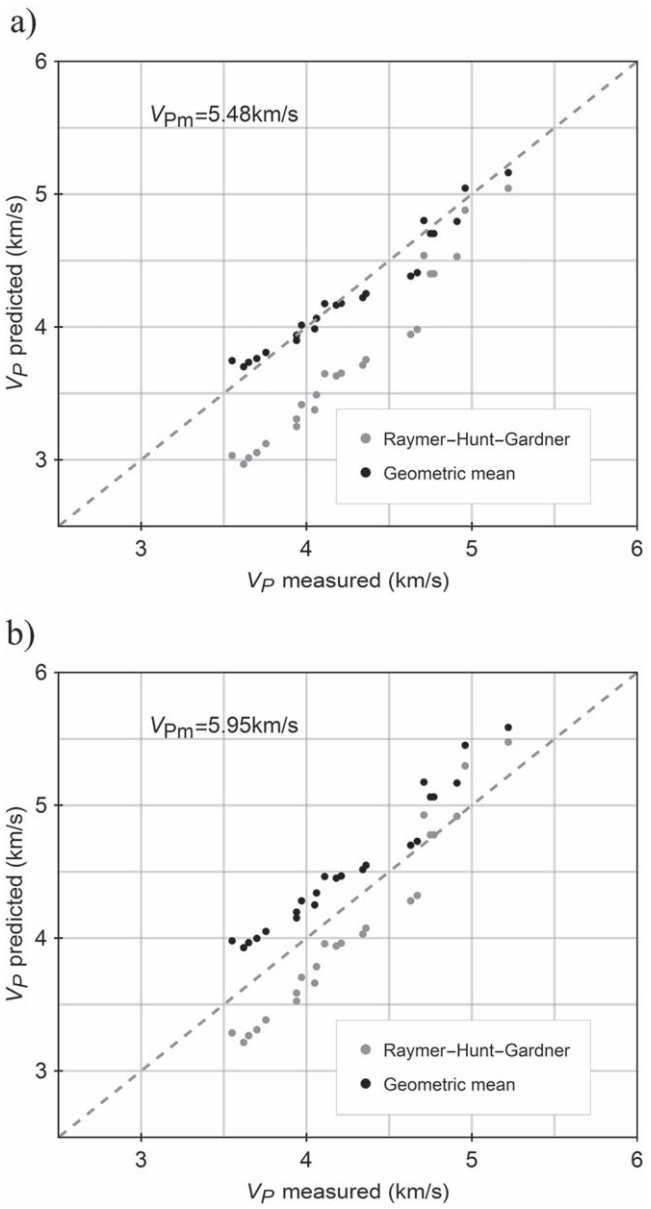

Figure 2. Comparing velocity-porosity models by the Raymer–Hunt–Gardner relation and the geometric mean. The sandstone data measured at 50 MPa by Han (1986) are used. Only samples with clay content less than 10% are used. (a) The lowest value of V_{Pm} suggested by Mavko *et al* (1998) for sandstone is used; (b) the highest value of V_{Pm} for sandstone is used.

relation can be expressed in the form

$$\frac{1}{V_P} = \frac{1-\phi}{V_{Pm}} + \frac{\phi}{V_{Pf}}, \tag{9}$$

where V_P, V_{Pm} and V_{Pf} are the P-wave velocities of the saturated rocks, of the mineral matrix, and of the pore fluid, respectively. For sandstones, V_{Pm} should be in the range from 5.48 to 5.95 km s^{-1} (Mavko *et al* 1998). Raymer *et al* (1980) suggested improvements to Wyllie's empirical velocity-porosity relation as follows:

$$V_P = (1-\phi)^2 V_{Pm} + \phi V_{Pf}\ ,\ \phi < 37\%, \tag{10}$$

$$\frac{1}{\rho V_P} = \frac{1-\phi}{\rho_m V_{Pm}{}^2} + \frac{\phi}{\rho_f V_{Pf}}, \quad \phi > 37\%, \tag{11}$$

where V_P, V_{Pm} and V_{Pf} are the P-wave velocities of the saturated rocks, of the mineral matrix, and of the pore fluid, respectively. The terms ρ, ρ_m and ρ_f are the densities of the saturated rocks, of the mineral matrix, and of the pore fluid, respectively. Here 37% is the critical porosity, above which the elastic properties are estimated by the Reuss bound. Equations (10) and (11) are called the Raymer–Hunt–Gardner relation and it is applied in the industry to estimate the porosity from measurements of seismic velocities and knowledge of the rock type and pore fluid content (Mavko *et al* 1998).

A velocity-porosity model can be directly derived from the power mean or geometric mean. From the geometric mean (equation (8)), we have

$$\rho V_P{}^2 = (\rho_m V_{Pm}{}^2)^{1-\phi}(\rho_f V_{Pf}{}^2)^{\phi}, \quad \phi < 37\%, \tag{12}$$

where the notations have same meanings as those in equations (10) and (11).

Using Han's data (1986), figure 2 shows a comparison of the Raymer–Hunt–Gardner model and the geometric mean in modeling the velocity-porosity relationship. In consideration of the assumption of uniform lithology and high effective pressure, only data points with a clay content less than 10% and measured at a differential pressure of 50 MPa are included. When a low value of 5.48 km s^{-1} is used for the mineral P-wave velocity, the velocity-porosity model by the geometric mean (equation (12) fit the data much better than the Raymer–Hunt–Gardner relation does. Relative to the measured P-wave velocity, the root mean square error (RMSE) is 0.547 km s^{-1} for the Raymer–Hunt–Gardner relation and the RMSE is 0.107 km s^{-1} for the model by the geometric mean. When a high value of 5.95 km s^{-1} is used for the mineral P-wave velocity, neither model fits the measured velocity-porosity trend very well. The RMSE is 0.299 km s^{-1} for the Raymer–Hunt–Gardner relation, and it is 0.300 km s^{-1} for the model by equation (12). When the Raymer–Hunt–Gardner model is used, the data points are separated into two trends at 4.75 km s^{-1} and it cannot fit the measured velocity regardless of whether a high or low V_{Pm} is used. Therefore, the velocity-porosity model using the geometric mean can be an improvement over the Raymer–Hunt–Gardner relation.

For some reservoirs, the variation of V_{Pm} may not be sufficient to describe the velocity-porosity relationship. In such a case, we may try using the velocity-porosity relation derived from the general mean in the form of equation (7), and that is,

$$\rho V_P{}^2 = ((1-\phi)(\rho_m V_{Pm}{}^2)^a + 1 - \phi(\rho_f V_{Pf}{}^2)^a)^{1/a}, \tag{13}$$

where a is the power parameter in a range of $[-1, 1]$, and the other notations have same meanings as those in equations (10) and (11). To model the effect of an individual mineral phase on the P-wave velocity, such as the clay mineral, a multiple-term power mean can be used to construct a velocity-porosity model similar to equation (13).

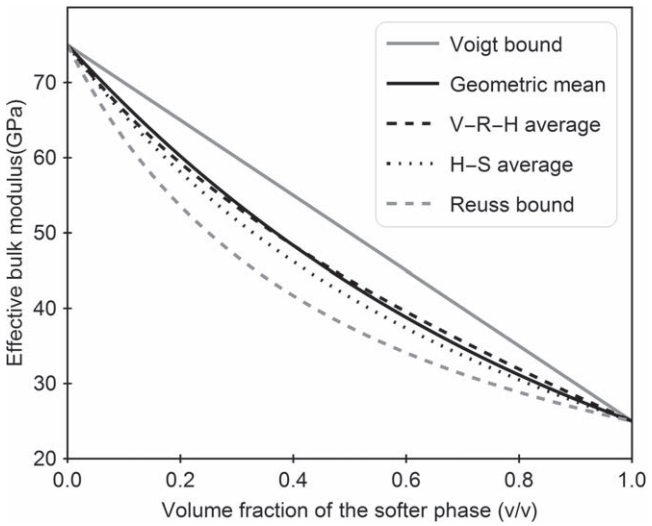

Figure 3. Comparison of V–R–H average, H–S average and weighted geometric mean in modeling the effective bulk modulus of the solid rock frame. The bulk modulus of the two mineral phases are 75 and 25 GPa, respectively.

Effective media and weighted geometric mean

In the rock physics study, such as Gassmann fluid substitution (Gassmann 1951), a regular task is to estimate the effective elastic moduli of the mineral matrix. Knowing the concentration of the mineral phases, the effective elastic moduli of the mineral matrix are primarily controlled by the texture of the rocks. It is difficult to definitely determine the effective elastic moduli because the texture is a qualitative description of the rock. Two procedures are often used to approximate the effective elastic moduli of the mineral matrix: Voigt–Reuss–Hill (V–R–H) average and Hashin–Shtrikman (H–S) average (Smith *et al* 2003). The Voigt bound and Reuss bound are the power means with power parameter equals to 1 and −1, respectively. The weighted geometric mean is a power mean with power parameter of 0 and the power mean is a monotonic function of the power parameter. It is natural to consider that the weighted geometric mean may be a good option to approximate the elastic moduli of the mineral matrix. Using elastic properties of calcite and clay as shown in table 1, figure 3 illustrates the differences in estimating the bulk modulus of the mineral matrix. For the mineral mixture of calcite and clay, the difference between the V–R–H average and the weighted geometric mean is small. When the concentration of the softer phase, the clay, is low, the geometric mean is slightly higher than the V–R–H average; when the concentration of the softer phase is high, the geometric mean is slightly lower than the V–R–H average. The maximum difference is 0.86 GPa, which may be not important for common applications. The H–S average is lower than both the V–R–H average and the geometric mean. The maximum difference between the H–S average and the geometric mean is 2.28 GPa for the mixture of calcite and clay. Selection of the optimum averaging scheme depends on the diagenesis of rock under study. Among the three average schemes, the geometric mean has the most simple and neat form.

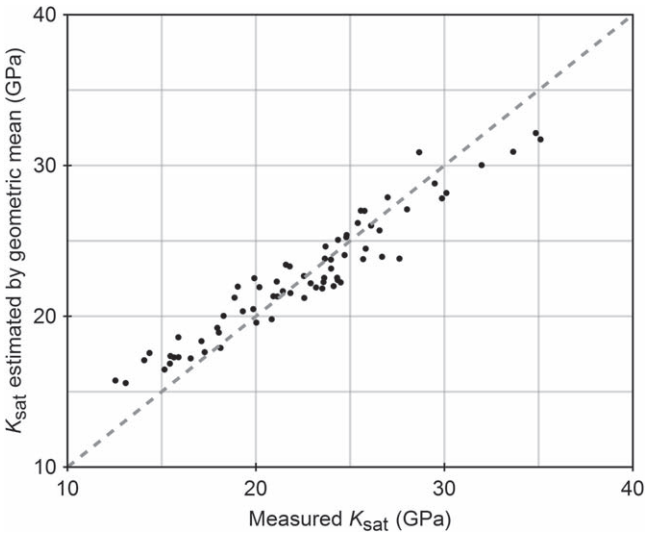

Figure 4. Estimation of the effective bulk modulus of saturated sandstones using weighted geometric mean. The laboratory ultrasonic measurement data are from Han (1986). Only data points measured at a differential pressure of 50 MPa are shown.

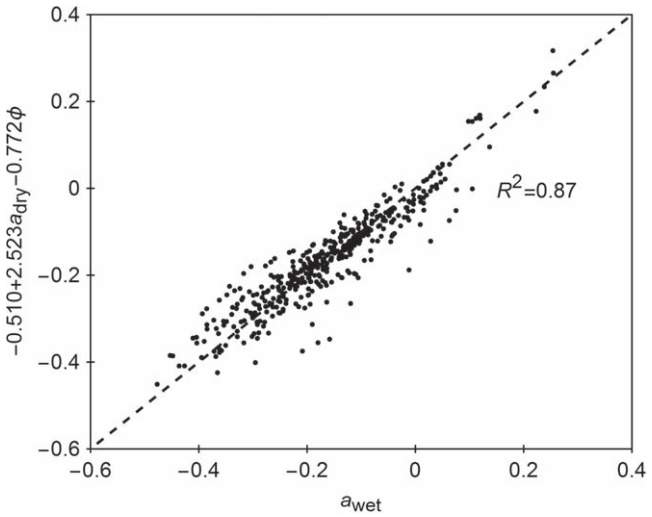

Figure 5. Correlation between the power parameter a_{wet} for fully brine-saturated rocks and a_{dry} for dry rocks.

Figure 4 shows the results of using the weighted geometric mean to estimate the effective bulk modulus of saturated sandstones. The data come from laboratory ultrasonic measurements by Han (1986). The 69 sandstone samples come from different places all over the world. Only data measured at a differential pressure of 50 MPa are shown. The effective bulk moduli estimated by the weighted geometric mean are slightly out of trend with the measured bulk moduli. For sandstones with lower bulk modulus, the estimation of the geometric mean is higher than the measurement; for sandstones with higher bulk modulus, the estimation of the geometric mean is lower than the measurement. Therefore, the power mean with $a = 0$ is not sufficient to characterize the properties of general reservoir rocks. We need to characterize the power parameter a for different rocks and for rocks saturated with different pore fluids.

The power parameter for dry and saturated rocks

From equation (7), if the rocks are dry, the P-wave modulus of pore fluids is zero. The power parameter a should be positive and in the range of (0, 1] for dry rocks because the negative power of zero is not rational. We drop the term with M_f in equation (7) and the effective P-wave modulus estimated by the power mean is in the form

$$M_{dry} = ((1 - \phi)M_m{}^{a_{dry}})^{1/a_{dry}}, \tag{14}$$

where the subscript 'dry' refers to dry rocks. If we have P-wave velocity measurement on dry rocks, then the power parameter a_{dry} for dry rocks can be calculated by

$$a_{dry} = \frac{\ln(1 - \phi)}{\ln M_{dry} - \ln M_m}. \tag{15}$$

If we have the P-wave velocity measurement on saturated rocks, a_{wet} can be inverted from equation (7) because it is the sole unknown variable. The relation between a_{dry} and a_{wet} can be set up if P-wave velocity measurements are made on rock samples under both dry and saturated conditions.

Figure 5 shows the correlation of a_{wet} with a_{dry} and porosity for sandstones. The sandstone data are from Han (1986), and they include measurements at different pressure conditions. The sandstone samples are collected all over the world. Some of them are clean and consolidated, such as the Berea sandstone, Fontainebleau sandstone; and some are less consolidated and have more clay content, such as those from the Gulf of Mexico. The porosity of the samples varies from 4% to 31%, and the clay content varies from 0% to 51%. In spite of the variety of the sandstone samples, the power parameters for fully brine-saturated sandstones are well correlated with the power parameters for the dry sandstones. This empirical relation may be applied to predict the saturation effect on P-wave velocity. From P-wave velocity measurements on dry rocks, a_{dry} can be estimated using equation (13). Using the empirical relation shown in figure 5,

$$a_{wet} = -0.510 + 2.523a_{dry} - 0.772\phi\ , \tag{16}$$

where the power parameter a_{wet} for saturated rocks can be estimated. Using equation (16), we can then predict the saturation effect on P-wave modulus. Figure 6 shows the prediction of the saturation effect on P-wave velocity for sandstone. The prediction results are compared to those from the commonly used Gassmann equations (Gassmann 1951),

$$\frac{K_{wet}}{K_m - K_{wet}} = \frac{K_{dry}}{K_m - K_{dry}} + \frac{K_f}{\phi(K_m - K_f)}, \tag{17}$$

$$G_{wet} = G_{dry}, \tag{18}$$

where G refers to the shear modulus. Here the prediction of

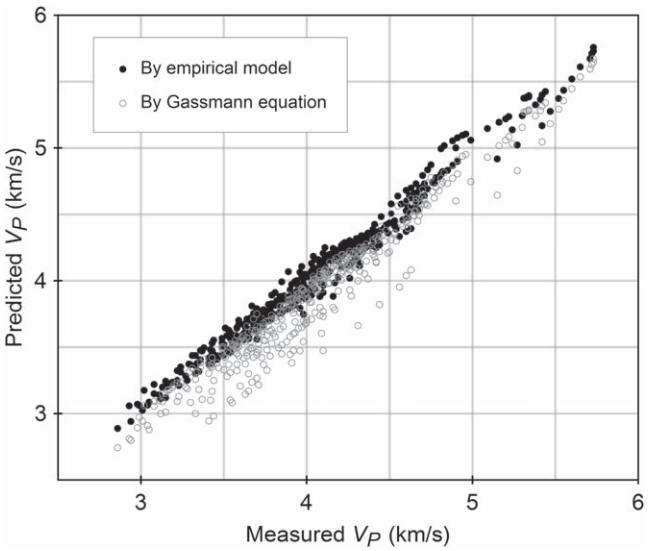

Figure 6. Prediction of the saturation effect on P-wave velocity of sandstone using the empirical model and the Gassmann equation. Han's data (Han 1986) are used. The RMSE for the empirical model and the Gassmann equation are 0.08 km s^{-1} and 0.21 km s^{-1}, respectively.

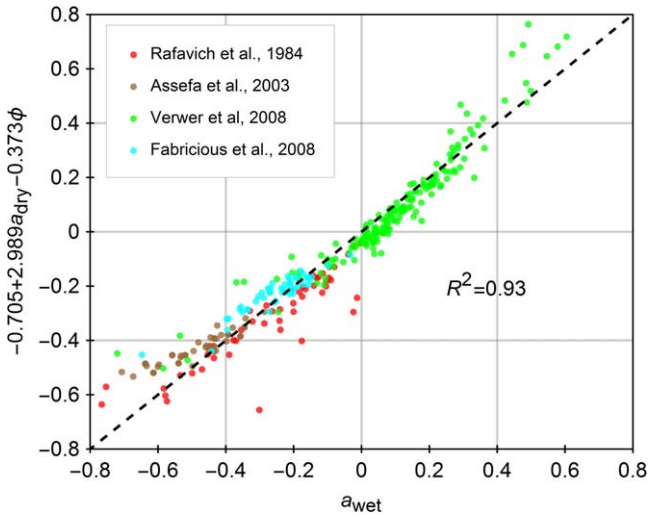

Figure 7. Correlation between the power parameter a_{wet} for fully brine-saturated carbonate rocks and a_{dry} for dry carbonate rocks.

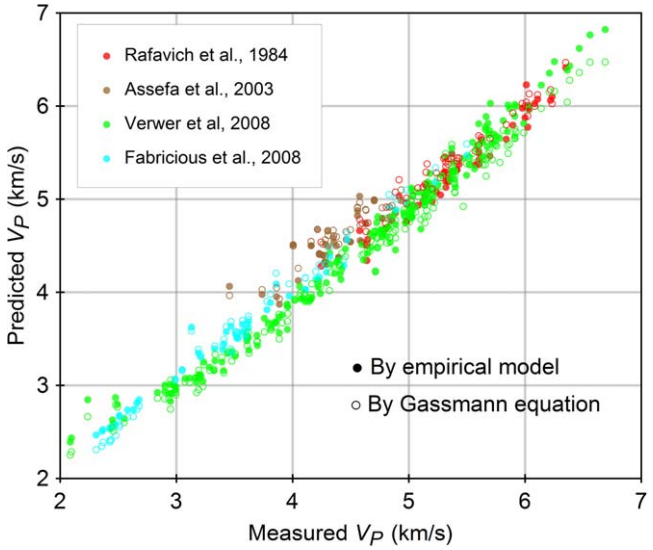

Figure 8. Prediction of the saturation effect on P-wave velocity of carbonate rocks using the empirical model in figure 7 and the Gassmann equation. The RMSE for the empirical model and Gassmann equation are 0.17 km s^{-1} and 0.19 km s^{-1}, respectively.

the saturation effect using the empirical relation is not a prediction in a strict sense because the same data for generating the empirical relation are used, but it still can demonstrate the potential utility of this methodology as a tool to study the saturation effect of pore fluids on seismic velocities. Figure 6 shows the prediction of the saturation effect on P-wave velocity for sandstone using the empirical model by equation (16) and the Gassmann equation, respectively. The root mean square error (RMSE) for the empirical model is 0.08 km s^{-1}, and the root mean square error for the Gassmann equation is 0.21 km s^{-1}. One advantage of the approach of fluid substitution using the empirical relation and the power mean is that shear velocity information is not needed. The greater prediction errors for the Gassmann equation are primarily caused by dispersion at low pressures when most of the cracks are open (Yan *et al* 2014). It does not necessarily mean that the Gassmann equation is a worse model. For a certain reservoir, if we have plenty of laboratory measurements on core plugs, a specified empirical relation similar to equation (16) can be set up to predict the fluid saturation effect, and its performance could be better than that of the Gassmann equation.

Figure 7 shows the correlation of a_{wet} with a_{dry} and porosity for carbonate rocks. The carbonate rock data sets are from Rafavich *et al* (1984), Assefa *et al* (2003), Verwer *et al* (2008), and Fabricius *et al* (2008). The carbonate rocks include limestone, dolomite, and chalk. The porosity of the carbonate rocks varies from 0.04 to 0.55. Compared to the sandstone dataset from Han (1986), the power parameter of the brine-saturated carbonates has a wider distribution. The empirical relations for sandstone and carbonates are similar, and the primary difference comes from the coefficient in the porosity term. The pore geometry of the carbonate rocks might be more complicated because of biogenic and chemical factors. Figure 8 shows the prediction of the saturation effect on P-wave velocity for carbonate rocks using the empirical relations and the Gassmann equation, respectively. The root mean square error (RMSE) for the empirical model is 0.17 km s^{-1} and the root mean square error for the Gassmann equation is 0.19 km s^{-1}. The results are similar. From figures 6 and 8, the saturation effect on P-wave velocities is more complicated for carbonate rocks than sandstone. The measured P-wave velocities are usually higher than those predicted by the Gassmann theory for sandstone, which is theoretically correct because the P-wave velocity predicted by the Gassmann theory is the lower bound. For carbonate rocks, it is not rare to see that the measured P-wave velocity is lower than that predicted by the Gassmann theory. This phenomenon cannot be explained by elastic theory. It might be caused

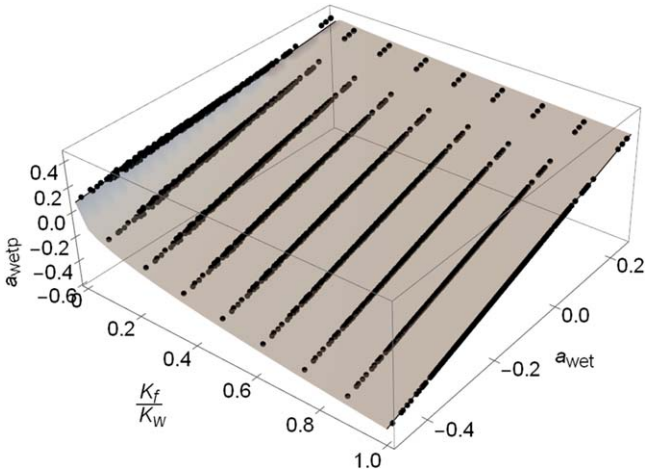

Figure 9. Variation of the power parameter when part of the brine is replaced by other pore fluid. Han's data (1986) and the Gassmann equation are used to produce the synthetic data of the power parameter. The gray surface shows the fitting of the synthetic data. The data points turn gray in color when they are under the gray surface.

by the chemical reaction between the water and the solid frame.

The power parameter for partially saturated reservoir rocks

From the previous study, the power parameter a for the dry rock is in the range (0, 1] and it is in the range [−1, 1] for fully saturated rocks. It is rational to believe that the power parameter will change with pore fluids. If the power parameter for the fully brine-saturated rocks is known, it may be useful to know how the power parameter will change when part of the brine is replaced by oil or gas. This information may be acquired by a large amount of laboratory measurements on partially saturated rocks although it is quite time-consuming and expensive. We did not make the measurements and no such data are available. Instead, the study is based on regular laboratory ultrasonic measurements on fully saturated rocks, and then the Gassmann equation in the following form is used to predict the elastic properties of partially saturated rocks:

$$\frac{K_{wet}}{K_m - K_{wet}} - \frac{K_w}{\phi(K_m - K_w)} = \frac{K_{wetp}}{K_m - K_{wetp}} - \frac{K_{fp}}{\phi(K_m - K_{fp})}, \tag{19}$$

where the subscript 'p' denotes properties related to partial saturation and K_w is the bulk modulus of the brine. The shear modulus is assumed to be independent of pore fluids. Finally, equation (7) is used to invert the power parameter for partially saturated rocks, which is denoted as a_{wetp}.

Figure 9 shows the cross plots between the power parameter a_{wet} for fully saturated sandstones and the power parameter a_{wetp} for the same set of sandstone samples that are partially saturated. The ultrasonic measurements on fully saturated sandstones are from Han (1986). No matter how much pore space is replaced by other pore fluids, the power parameter for partially saturated rocks is well correlated to the power parameter for fully saturated rocks, and the correlation is stronger when the difference in bulk modulus of the pore fluids is smaller. Based on data shown in figure 9, a regression relation between the power parameter for partially saturated rocks and the power parameter for fully saturated rocks is given by

$$a_{wetp} = 0.275 - 0.276\sqrt{\frac{K_{fp}}{K_w}} + \left(0.334 + 0.648\sqrt[4]{\frac{K_{fp}}{K_w}}\right) a_{wet}\,. \tag{20}$$

The regression coefficient R^2 for the above formula is 0.998, and the root mean square error is 0.007. Most of the scattering occurs when the brine is completely replaced by gas or vacuum. In the subsurface conditions, a complete fluid replacement can rarely happen because of the connate water and complicated pore geometry (Yan and Han 2016). For a certain porous rock, the change in the elastic properties due to different pore fluids can be represented by variation of the power parameter. Therefore, it is possible to design a fluid substitution scheme based on the empirical relation similar to equation (20).

P-wave modulus fluid substitution

A practical problem which often arises in the application of fluid substitution is that we wish to estimate the change of V_P but the shear velocity is usually unknown *in situ* (Mavko *et al* 1995). To get around this problem, Mavko *et al* (1995) developed an approximate method of fluid substitution. The approximate formula is in the same form as the Gassmann equation except that the P-wave moduli are replaced by bulk moduli:

$$\frac{M_{wet}}{M_m - M_{wet}} - \frac{M_w}{\phi(M_m - M_w)} = \frac{M_{wetp}}{M_m - M_{wetp}} - \frac{M_{fp}}{\phi(M_m - M_{fp})}, \tag{21}$$

The P-wave modulus equals the bulk modulus for most fluids with negligible shear modulus.

P-wave modulus fluid substitution can also be realized by the power mean. If a_{wet} for fully brine-saturated rocks is known, the regression formula in equation (20) and the power mean can be used for empirical fluid substitution. The P-wave velocity change due to fluid replacement is determined from the variation of the power parameter of the power mean. The variation of the power parameter is described by equation (20). Using Han's data (Han 1986), figure 10 shows a comparison of these two fluid substitution methods when the bulk modulus of the pore fluid changes from 2.3 GPa to 0.09 GPa and from 2.3 GPa to 1.16 GPa, respectively. The

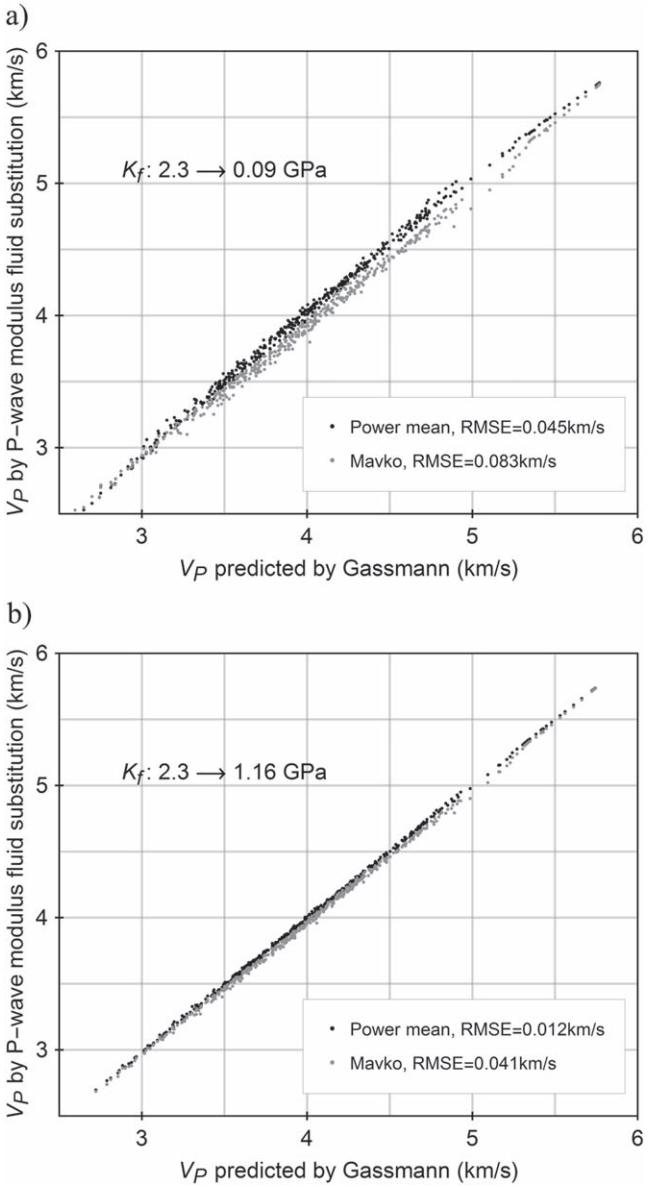

Figure 10. Comparing P-wave modulus fluid substitutions using the power mean and the Gassmann-like fluid substitution formula by Mavko *et al* (1995): (a) gas replaces brine, (b) oil replaces brine.

pore fluid changes simulate the brine of a subsurface reservoir rock being replaced by gas and oil, respectively, with only connate water left. The average saturation effect on P-wave velocity for all the data points due to gas replacing brine is $0.094\,\mathrm{km\,s^{-1}}$, and it is $0.053\,\mathrm{km\,s^{-1}}$ for the case of oil replacing brine. If the saturation effect predicted by the Gassmann equation is used as a standard, the RMSEs of the saturation effects on P-wave velocity predicted by the method proposed by Mavko *et al* (1995) are $0.083\,\mathrm{km\,s^{-1}}$ and 0.041 when the brine is replaced by gas and oil, respectively. The prediction error is very close to the saturation effect itself. Using the P-wave modulus fluid substitution with power mean, the prediction errors can be much lower than the saturation effect. The method of P-wave modulus fluid substitution with power mean has the potential to improve the results of saturation effect prediction if local calibration data are available. However, the method proposed by Mavko *et al* (1995) is still a robust and practical method because it does not need local calibration and the uncertainty in determining the seismic velocity is usually greater than the prediction error.

Discussions

The power mean has a neat form and the power parameter is the most critical variable. In its application to rock physics studies, the power mean has definite physical meaning when the power parameter takes values of -1 or 1. Since the power parameter -1 denotes the iso-stress state and the power parameter 1 denotes the iso-strain state, the power parameter varying from -1 to 1 can be interpreted as the gradual evolution of the stress state of a rock from the iso-stress state to the iso-strain state during formation and diagenesis of the rock. The exploration of the physical significance of the properties of a rock being well described by the geometric mean may be deserving of further effort. The power mean with other power parameter values should be treated as a phenomenological model, not a theoretical model. The power mean of P-wave moduli in the form of equation (7) is a convenient treatment in consideration of practical applications, and it cannot be derived from the power mean of bulk moduli and the power mean of shear moduli. It is preferable that the empirical relations brought up in this study are calibrated to local reservoirs before applications. The potential applications in rock physics of the power mean should not be limited to what we have demonstrated in this study. For example, the porosity of deep crust rocks is usually negligible, and the power mean can be a convenient and effective tool to model the relationship between the mineral compositions and seismic velocities.

Conclusions

In the terminology of statistics, the Voigt bound and the Reuss bound are special cases of the weighted power mean with the power parameter being 1 and -1, respectively. The power mean becomes the geometric mean when the power parameter is 0. With its neat form and flexibility, the power mean can be a very useful tool for modeling the elastic properties of rocks. We have demonstrated the potential applications of the power mean in studies of the velocity-porosity relationship, effective media, and the saturation effect of pore fluids on seismic velocities.

Acknowledgments

We would like to thank the Fluid and DHI consortium sponsors for the financial support.

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