Theoretical validation of fluid substitution by Hashin-Shtrikman bounds

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Summary

In this study, we find that variation of Hashin-Shtrikman bounds caused by pore fluid change is governed by the Biot-Gassmann theory. Also, we have shown that it is generally valid to assume that the relative position of the effective moduli of a rock within the Hashin-Shtrikman bounds is not affected by pore fluid change. Thus Hashin-Shtrikman bounds can be used as a valid fluid substitution tool. Comparing with Gassmann fluid substitution, the velocity predicted by Hashin-Shtrikman bounds generally fits better with the measured data since part of the dispersion is included, and it can be used as a more general pore infill substitution tool.

Introduction

Hill (1963) suggested that an arithmetic average of the Voigt-Reuss bounds might be a good estimate for the effective moduli of a two phase media. Marion and Nur (1990) extended Hill's idea and stated that the effective moduli is better approximated by a weighted average between any upper bound (M^+) and lower bound (M^-):

$$M = M^- + w\left(M^+ - M^-\right) \tag{1}$$

They define w as a measure of the average stiffness of the pore space. When pore fluid changes, the bounds might also change. But without any theoretical justification, they simply assume that weighting factor keeps constant. By assuming constant weighting factor, the effective moduli with new pore fluids can be calculated from the old effective moduli, old bounds and new bounds. This method of fluid substitution is called Balance Average Method (BAM) by Marion and Nur (1990). Also they found that if the Hashin-Shtrikman bounds are used, effective moduli predicted by the BAM method can fitter better with measured data than those predicted by Gassmann equations.

It must be pointed out the definition of w as a measure of the average stiffness of pore space and the assumption of constant w (not affected by pore fluid change) are contradictory: It is well known the stiffness of pore space is affected or even sensitive to pore fluid compressibility (Brown and Korringa, 1975 and Thomsen, 2010).

Deficient of theoretical justification the BAM method is rarely used in the industry. The primary goal of this paper is to theoretically prove that the BAM method using Hashin-Shtrikman can be a valid fluid substitution tool.

Consistency with Biot-Gassmann theory

The Hashin-Shtrikman bounds for bulk modulus and shear modulus are given by (Hashin and Shtrikman, 1963):

$$K^{HS\pm} = K_1 + \frac{f2}{\dfrac{1}{K_2 - K_1} + \dfrac{3f1}{3K_1 + 4G_1}} \tag{2}$$

$$G^{HS\pm} = G_1 + \frac{f2}{\dfrac{1}{G_2 - G_1} + \dfrac{6f1(K_1 + 2G_1)}{5G_1(3K_1 + 4G_1)}} \tag{3}$$

Assuming the pore space is empty with K_f=0 and G_f=0, from the Hashin-Shtrikman upper bound we have

$$K_{dry}^{HS+} = K_m + \frac{\phi}{\dfrac{1}{0 - K_m} + \dfrac{3(1-\phi)}{3K_m + 4G_m}} \tag{4}$$

Using the lemma if *a/b=c/d* and then *a/(b-a)=c/(d-c)*, equation (4) can be rewritten as :

$$\frac{K_{dry}^{HS+}}{K_m - K_{dry}^{HS+}} = -\frac{K_m}{\phi}\left(-\frac{1}{K_m} + \frac{3(1-\phi)}{3K_m + 4G_m}\right) - 1 \tag{5}$$

Assuming pore spaces are saturated, from Hashin-Shtrikman upper bound:

$$K_{sat}^{HS+} = K_m + \frac{\phi}{\dfrac{1}{K_f - K_m} + \dfrac{3(1-\phi)}{3K_m + 4G_m}} \tag{6}$$

Similarly it can be rewritten as

$$\frac{K_{sat}^{HS+}}{K_m - K_{sat}^{HS+}} = -\frac{K_m}{\phi}\left(\frac{1}{K_f - K_m} + \frac{3(1-\phi)}{3K_m + 4G_m}\right) - 1 \tag{7}$$

Let equation (7) subtract (5) and after simplification, we can get

$$\frac{K_{sat}^{HS+}}{K_m - K_{sat}^{HS+}} = \frac{K_{dry}^{HS+}}{K_m - K_{dry}^{HS+}} + \frac{K_f}{\phi\left(K_m - K_f\right)} \tag{8}$$

Thus the Hashin-Shtrikman upper bound for bulk modulus is consistent with Biot-Gassmann theory. Similarly, for the Hashin-Shtrikman lower bound of bulk modulus, we can get

$$\frac{K_{sat}^{HS-}}{K_m - K_{sat}^{HS-}} = \frac{K_f}{\phi\left(K_m - K_f\right)} \tag{9}$$

Hashin-Shtrikman Bounds

This is actually Gassmann equation for special case when dry bulk modulus equals to zero, so that the Hashin-Shtrikman low bound for bulk modulus is also consistent with Biot-Gassmann theory.

If we assume shear modulus of pore fluid is zero, then pore fluid has no effect on Hashin-Shtrikman upper bound or lower bound for shear modulus.

In summary, the Hashin-Shtrikman bounds are consistent with Biot-Gassmann theory. Similar to Gassmann equation for a rock saturated with different pore fluids (Mavko, and et. al., 1998), it is straightforward from above derivation that we can have

$$\frac{K_{sat1}^{HS+}}{K_m - K_{sat1}^{HS+}} - \frac{K_{f1}}{\phi(K_m - K_{f1})} = \frac{K_{sat2}^{HS+}}{K_m - K_{sat2}^{HS+}} - \frac{K_{f2}}{\phi(K_m - K_{f2})} \quad (10)$$

$$\frac{K_{sat1}^{HS-}}{K_m - K_{sat1}^{HS-}} - \frac{K_{f1}}{\phi(K_m - K_{f1})} = \frac{K_{sat2}^{HS-}}{K_m - K_{sat2}^{HS-}} - \frac{K_{f2}}{\phi(K_m - K_{f2})} = 0 \quad (11)$$

Normalized stiffness

The effective moduli of natural porous rock usually lie within the relative Hashin-Shtrikman upper bound and lower bound. As shown in Fig. 1, if the pore fluid 1 in a rock is replaced by pore fluid 2, then both the Hashin-Shtrikman bounds of bulk modulus and the effective bulk modulus of the saturated rock will change. To describe the relative position of the effective moduli of a rock within corresponding Hashin-Shtrikman bounds, we define a parameter y_K as:

$$y_K = \frac{K - K^{HS-}}{K^{HS+} - K^{HS-}} \quad (12)$$

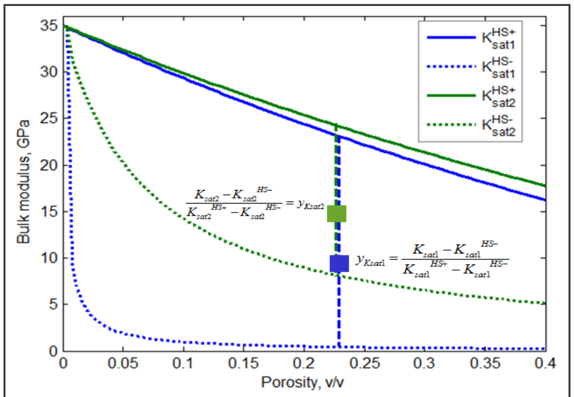

Figure 1: Sketch of variation of Hashin-Shtrikman bounds and normalized stiffness caused by pore fluid change

We call it normalized bulk stiffness of a mixture of a certain composition. Similarly we can define the normalized shear stiffness as:

$$y_G = \frac{G - G^{HS-}}{G^{HS+} - G^{HS-}} \quad (13)$$

Here the normalized stiffness is intrinsically the weighting coefficient used in BAM method and ranges between 0 and 1. For a rock of certain composition, its Hashin-Shtrikman bounds are constants; and the relative position of the effective moduli of a rock within the bounds is determined by the digenesis (sorting, grain contact, cementing, and pore geometry and et. al.) and stress state of the rock. So it has important physical meaning.

We have shown that variation of Hashin-Shtrikman bounds caused by pore fluid change is governed by Biot-Gassmann theory. For a real rock whose effective moduli lying within the bounds, we don't know how the normalized stiffness changes with pore fluid, but we can assume that it is also governed by Bio-Gassmann theory and then study how it varies with pore fluid. Here we consider the extreme case of pore fluid change: from dry rock to fully saturated rock.

For a dry rock, the low bound of bulk modulus is zero, so that the normalized bulk stiffness of dry rock is

$$y_{Kdry} = \frac{K_{dry}}{K_{dry}^{HS+}} \quad (14)$$

When this rock is fully saturated, we have

$$y_{Ksat} = \frac{K_{sat} - K_{sat}^{HS-}}{K_{sat}^{HS+} - K_{sat}^{HS-}} \quad (15)$$

Let equation (15) minus equation (14) and then replace the saturated bulk moduli with dry bulk moduli, fluid bulk modulus (K_f) and matrix bulk modulus (K_m) using the Gassmann relations, and after simplification we can get

$$y_{Ksat} - y_{Kdry} = -\frac{K_{dry}}{K_{dry}^{HS+}} \cdot \frac{K_{dry}^{HS+} - K_{dry}}{K_m - K_{dry}} \cdot \frac{1}{1 + \phi \frac{(K_m/K_f) - 1}{1 - (K_{dry}/K_m)}} \quad (16)$$

By analyzing the physical meanings and relations of the parameters in the right side of equation (16), we can see that each of the three terms ranges between 0 and 1; usually the cubic power of a quantity ranging from 0 and 1 should be much small relative to 1. There is a minus sigh ahead of these three terms. So from above analysis we can see that

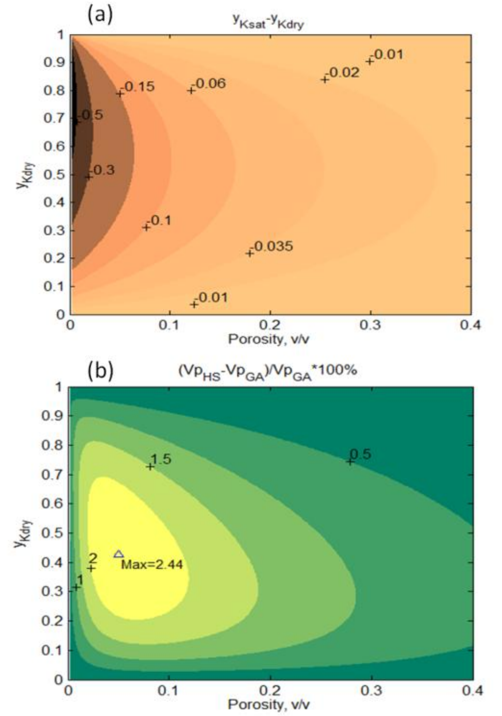

Figure 2: Sensitivity analysis

when a dry rock is fully saturated, the normalized bulk stiffness will slightly decrease. If we make an approximation by assuming the normalized stiffness is constant, then the P-wave velocity of saturated rock will be slightly overestimated comparing to that prediction by Biot-Gassmann theory.

Sensitivity analysis

From the third term of the right side of equation (16), the higher the porosity, K_m and the ratio of K_m over K_f, the smaller is change of the normalized stiffness. The effect of K_{dry} is not clear from the formulation. To have direct understanding of the effect of pore fluids on variation of the normalized stiffness, sensitivity analysis was made using reasonable rock parameters, as shown in Figure 2. Here with shaly sandstone in mind, we select Km=Gm=35GPa, grain density 2.63 g/cc, fluid bulk modulus 2.2 GPa and fluid density 1.0 g/cc. The panel (a) shows the sensitivity of variation of normalized stiffness to porosity and the original dry rock stiffness, if the porosity is higher than 10%, the change of normalized stiffness caused by pore fluid can be negligible.

In panel (b), we assume that pore fluid has no effect on the normalized stiffness and then use Hanshin-Shtrikman bounds for fluid substitution and the calculated velocity is denote as Vp_{HS}. The predicted P-wave velocity by Gassmann fluid substitution is denoted as Vp_{GA}. From panel (b) we can see that velocity predicted by Hashin-Shtrikman bounds is always slightly higher than that predicted by Gassmann equation, the maximum velocity difference occurring around porosity of 5% is less than 2.5%. So even for dense rock, the velocity error caused by assumption of constant normalized stiffness is not significant.

Example

We use Han's data (1986) to demonstrate the applicability of Hashin-Shtrikman bounds on fluid substitution. The general form of Hashin-Shtrikman bounds by Berryman (1995) is used for calculation since the actual rock is usually composited of more than two components. Figs. 3 to 7 show the result. For Figs 3 to 6, only data measured at differential pressure of 50 MPa is used.

Fig. 3 shows the correlations between normalized stiffness of dry and saturated rock, and between normalized stiffness of dry rock and the stiffness predicted by Biot-Gassmann theory. From Figure 3, most of the data points distributed

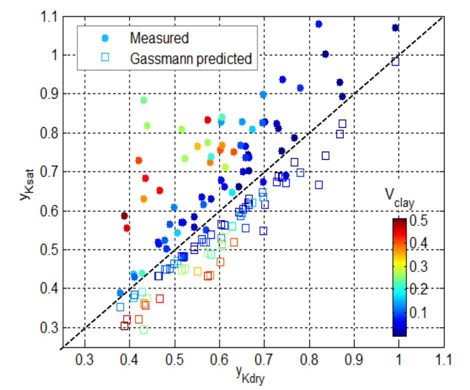

Figure 3: Comparing of normalized bulk stiffness of dry rock, saturated rock and Gassmann predicted

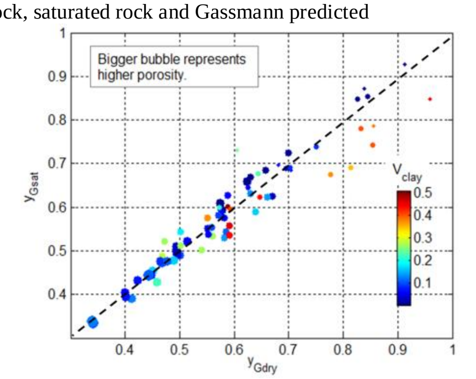

Figure 4: Comparing of normalized shear stiffness of dry and saturated rock.

Hashin-Shtrikman Bounds

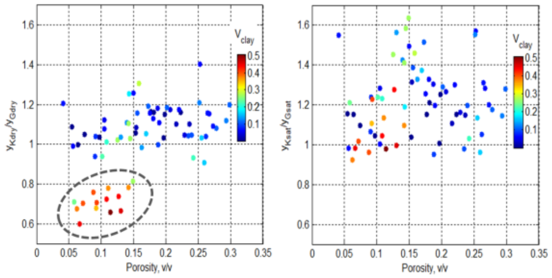

Figure 5: Clay swelling/shrinkage effect outlined by interpretation of normalized stiffness

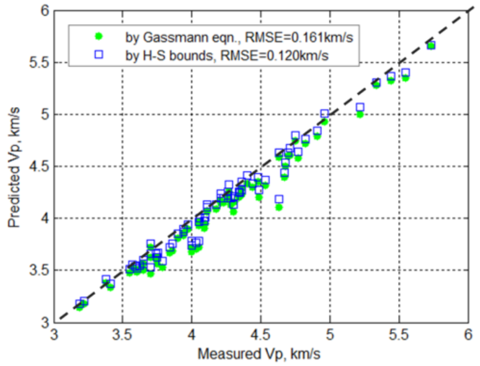

Figure 6: Comparing fluid substitutions by Gassmann and by Hashin-Shtrikman bounds respectively

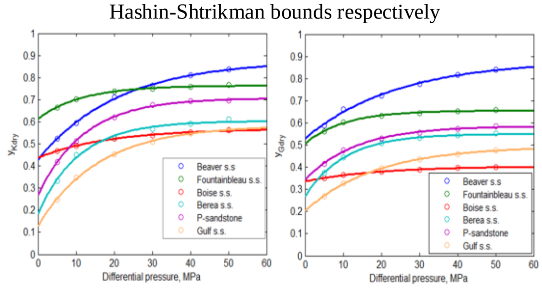

Figure7: Stress effect on normalized stiffness

along the perfect line, which means that the saturation does not have significant effect on normalized bulk stiffness, but for some core samples, the saturated bulk stiffness is much higher than the dry bulk stiffness; this might be caused by clay swelling effect or dispersions. The normalized stiffness predicted by Biot-Gassmann theory is always lower than that estimated from the measured data. The results comply with the theoretically analysis we made in previous section. Thus fluid substitution by Hashin-Shtrikman bounds with assumption of constant normalized bulk stiffness will generally fit better with measured data than Biot-Gassmann theory.

Fig. 4 shows the correlation between shear stiffness of dry and saturated rocks. As theoretically predicted the normalized stiffness is basically not affected by pore fluids. But for some core samples of shaly sandstone, there is obvious shear weakening phenomena. Combined with Fig. 5, the clay swelling effect outlined by the dashed circle is more easily identified and understood. It should be noted that there are many different types of clay minerals and the swelling behaviors are very different.

In Fig. 6, we compare the P-wave velocity predicted by Gassmann's equations and Hashin-Shtrikman bounds respectively using measured ultrasonic velocity. It can be seen that generally the P-wave velocity predicted by Hashin-Shtrikman bounds fit better with the measured data. This is because part of the dispersion effect is included in the velocity predicted by Hashin-Shtrikman bounds while velocity predicted by Biot-Gassmann theory is a low limit. Again, the comparison results comply with our previous theoretical analysis.

Fig. 7 shows stress effect on normalized stiffness for some of the core samples cited frequently in the literatures. The data are fitted with the general stress effect model (Yan and Han, 2007). It can be seen that the normalized stiffness might be very sensitive to stress effect. We have shown that the normalized stiffness is not sensitive to pore fluid (saturation) for porous reservoir rock. Thus this might be an import feature to discriminate saturation effect and stress effect for time lapse seismic study.

As shown in Marion and Nur's (1991) study, one potential application of fluid substation by Hashin-Shtrikman bounds is on heavy oil sands. The concept of "dry rock" is fuzzy when the pore infill is semi-liquid heavy oil. It is possible to model the normalized stiffness as function of temperature, stress and frequency, and then the P-wave velocity as function of these parameters can be modeled. We will discuss this problem especially in another paper.

Conclusions

The Hashin-Shtrikman bounds are consistent with Biot-Gassmann theory and they can be used as a valid fluid substitution tool. The normalized stiffness of a rock is not sensitive to saturation effect, but can be very sensitive to stress effect.

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EDITED REFERENCES

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